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Robustness of T Scores when the true score is considered As a Random Variables

Suraksha Pal and Sarika Sharma
Department of Education,
C.C.S*University, Meerut – 250 005 (India)

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Abstract:

Various testing situations have been analysed in the Bayesian setup in which updating the prior variations with experimental data has been the main concern. Here, it is recognized that prior variations do have an impact on the basic distribution of raw scores. Following the concept, the present study deals with the development of statistical methods useful for updating the basic distribution in view of the prior variations in true scores. Further, this updated basic distribution is used to analyze the sensitivity aspect of T-scores when the true score is considered as a random variable.

1.0 Introduction:

Suppose a standardized test in some subject be administered to a person. Let X be the random variable denoting the raw score of the person in the test. Here, the distribution of X is taken as normal with mean μ and variance σ^2, μ and σ^2 both being known constants. Obviously, μ stands for person's true score, a measure of person's true ability. Now, in a situation when either the same test or its parallel form is administered to the person in varying environmental conditions, resulting inculcation of measurement errors that might creep in due to the subjectivity arising from sampling of contents, sampling of objectives within the content, testing situations, inter- and intra-examiner variability, etc., the assumption regarding person's constant true score seems to be unrealistic and restrictive. To overcome this difficulty, the person's true score, μ, be considered as a random variable and the uncertainty about the value of μ be modelled by assigning a prior probability distribution to it. More so, testing with standardized tests is a continuous valuation process and, as such, a strong prior information representing variations in μ may be available. Following the concept, the study in [Novick & Jackson, 1974] has given the logical basis of Bayesian setup in which updating priors with experimental data in the form of posterior distributions has been the main concern. This posterior distribution is the main tool in the Bayesian setup. However, in the Bayesian framework, it should be recognized that

*Formerly Meerut University, Meerut.

random parametric variations as represented by priors do have an impact on the basic distribution of raw scores and, therefore, updating the basic distribution in respect of prior variations is another important aspect for analyzing the testing characteristics in the changed scenario.

In view of the above, the present study considers the development of statistical methods useful for analyzing the impact of prior variations on the basic distribution of raw scores. This updated basic distribution has been used for analyzing the sensitivity of T- scores when prior variations in true scores are suspected. The extent or intensity of the sensitivity of T-scores in the changed scenario is measurable in terms of a relative measure of dispersion called the coefficient of variations in the corresponding situations Theoretical results have been highlighted with numerical examples.

2.0 Notations and Assumptions:

X	: Random variable denoting the raw score of an individual in a test.
T	: T- Score
N (μ, σ^2)	: Normal distribution with mean μ and variance σ^2.
p.d.f.	: Probability density function
C.V.	: Co-efficient of Variation
X ~ N (μ, σ^2)	: Random variable X is normally distributed with mean μ and variance σ2.

It is assumed that –

(a) An individual's raw scores, X, on a standardized test is distributed as N (μ, σ_1^2), where μ and σ_1^2 are considered as known constants. The p.d.f. of the random variable X is

$$f_1(x,\mu,\sigma_1)=\frac{1}{\sigma_1\sqrt{2\pi}}\exp\left\{-\frac{1}{2}\left(\frac{x-\mu}{\sigma_1}\right)^2\right\} \;;\; -\infty< x,\, \mu<\infty,\ \sigma_1>0 \qquad \ldots (1)$$

Here, $E(X) = \mu$, $V(X) = \sigma_1^2$, $C.V. = \left(\frac{\sigma_1}{\mu}\right)\times 100$

Let us call the distribution of X in (1) as the basic distribution of raw scores.

(b) The true score, μ, is also a random variable following N (θ, σ_2^2) with p.d.f.

$$g_1(\mu,\theta,\sigma_2)=\frac{1}{\sigma_2\sqrt{2\pi}}\exp\left\{-\frac{1}{2}\left(\frac{\mu-\theta}{\sigma_2}\right)^2\right\} \;;\; -\infty< \mu,\, \theta<\infty,\ \sigma_2>0 \qquad \ldots (2)$$

Here, σ_1 is assumed to be known.

$E(\mu) = \theta$, $V(\mu) = \sigma_2^2$

3.0 Statistical Background:

(a) In view of (1) and (2), the compound distribution [Johnson & Kotz, 1969] of the random variable X can be obtained as

$$f_2\left(x,\theta,\sigma^2\right)=\int_{-\infty}^{\infty} f_1(x,\mu,\sigma_1)\, g_1(\mu,\theta,\sigma_2)\, d\mu$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \exp\left\{ -\frac{1}{2}\left(\frac{x-\theta}{\sigma}\right)^2 \right\} \; ; \; -\infty < x, \theta < \infty, \; \sigma > 0 \qquad \ldots (3)$$

It is notable here that the distribution of X in (3) is also N (θ, σ^2), where $\sigma^2 = \sigma_1^2 + \sigma_2^2$

Here, for this distribution, we have

$$E(X) = \theta, \qquad V(X) = \sigma^2, \qquad C.V. = \left(\frac{\sigma}{\theta}\right) \times 100$$

(b) T- scores are normalized scores with mean 50 and variance 100 (or standard deviation 10). Also the relationship between normally distributed raw scores and T- scores is linear.

In the compounding process, we observed that the variations in μ get neutralized by taking expectation over the function $f_1(x, \mu, \sigma_1)$ with respect to μ (for fixed X). Consequently, the distribution of X in (3) can be viewed as an updated basic distribution or the predictive distribution [Novick & Jackson, 1974] of the raw score X in respect of variations in true score μ, as represented in (2), have been taken into consideration.

Thus, in the process, one gets two distributions of the raw scores as given in (1) and (3) respectively. These distributions are useful for analyzing the sensitivity of T- scores when μ is considered as a random variable. In this concern, we consider the following two specific situations –

(1) When the basic distribution of raw score X, as given in (1), is used in the analysis. Here the true score μ is considered as a constant.

(2) When the updated basic or predictive distribution of X, as given in (3), is used in the analysis. Here, the distribution of X in (3) considers μ as a random variable.

4.0 T- Score when μ is considered as constant:

For defining T- scores in this situation we use the basic distribution of X as given in (1).

Here $X \sim N(\mu, \sigma_1^2)$ and $T \sim N(50, 100)$

Thus,

$$\left(\frac{X-\mu}{\sigma_1}\right) \sim N(0,1) \text{ and } \left(\frac{T-50}{10}\right) \sim N(0,1)$$

Therefore,

$$\left(\frac{T-50}{10}\right) = \left(\frac{X-\mu}{\sigma_1}\right)$$

or

$$T = 50 + 10\left(\frac{X-\mu}{\sigma_1}\right) \qquad \ldots (4)$$

The formula in (4) is useful in transforming raw scores to their corresponding T- score for known values of μ and σ_1.

5.0 T- Scores when μ is considered as a random variable:

For defining T- scores in this situation we make use of the updated basic distribution or predictive distribution of X in (3). Just for making a differentiation in the present situation, we denote the random variable X and T - in section 4.0 as X' and T' - respectively.

Thus, $X' \sim N(\theta, \sigma^2)$ and $T' \sim N(50, 100)$

Therefore,

$$\left(\frac{X'-\theta}{\sigma}\right) \sim N(0,1) \text{ and } \left(\frac{T'-50}{10}\right) \sim N(0,1)$$

Therefore,

$$\left(\frac{T'-50}{10}\right) = \left(\frac{X'-\theta}{\sigma}\right)$$

or

$$T' = 50 + 10\left(\frac{X'-\theta}{\sigma}\right) \qquad \ldots (5)$$

The transformation in equation (5) is useful for converting the raw scores X' to their corresponding T' - scores when μ is considered a random variable.

6.0 Discussion:

It is now observed that –

(a) The basic distribution of the raw score, X, is given in (1). Here, the true score μ is assumed to be constant. The equation (4) in section 4.0 can now be used for transforming raw scores X to T- scores when μ and σ_1 are known.

(b) In view of prior variations in μ, the basic updated distribution of X' is given in (3). The equation (5) in section 5.0 is thus useful for transforming X' to T' - when μ is considered as a random variable.

(c) For introducing statistical validity in the comparison of T- scores and T' - scores obtained by using the respective transformations in (4) and (5), the reference points are chosen so that E(X) for the distributions in (1) and (3) are equal, i.e.,

$$\mu = E(\mu) = \theta \qquad \ldots (6)$$

Equation (6) clearly states that the mean of the random variable μ is equal to μ (when considered as constant).

In view of the relationship in (6), the relative measure of dispersion, i.e., the respective C.V's of the basic and updated basic distributions in (1) and (3) can be viewed as error of measurement for the estimated points of T - and T' - scores in the corresponding situations. Further, the percentage increase or decrease in C.V's in the two situations can be viewed as the measure of intensity of the sensitivity of T- scores when true score μ varies randomly.

7.0 An Example:

Here, we consider the following two data sets in the analysis -

I data set - $\left[X \sim N(\mu = 40, \sigma_1 = 4), \mu \sim N(\theta = 45, \sigma_2 = 3) \text{ and consequently } X' \sim N(\theta = 45, \sigma = 5)\right]$

II data set - $\left[X \sim N(\mu = 40, \sigma_1 = 4), \mu \sim N(\theta = 35, \sigma_2 = 3) \text{ and consequently } X' \sim N(\theta = 35, \sigma = 5)\right]$

As stated, for statistical validity of making comparisons between T- scores in the two situations, we make use of transformations in (4) and (5), considering the I data set the values for T - and T' - scores for varying values of $\mu = \theta$ [Equation (6)] and for some fixed values of $(X = X')$ have been summarized in Table – 1. For still better analysis, the values in Table – 1 are also plotted in Figs. 1- 4.

The variations in T - and T' - can also be analyzed in another data set-up. In this case the transformations in (4) and (5) are used to study variations in T - and T' - for varying values of $X = X'$. The variations for both the data sets have also been listed in Table – 2. The values in Table – 2 have also been plotted in Figs. – 5 and 6.

Further, for comparing the error of measurement involved in the estimated points of T - and T' - scores, we calculate –

(a) The Coefficient of Variation for the distribution in (1) as

$$C.V. = \left(\frac{\sigma_1}{\mu}\right) \times 100 = \left(\frac{4}{40}\right) \times 100 = 10.0 \qquad \ldots (7)$$

(b) The Coefficient of Variation for the distribution in (3), which has been updated in respect of prior in (2), as

$$C.V. = \left(\frac{\sigma}{\theta}\right) \times 100 = \left(\frac{5}{45}\right) \times 100 = 11.11 \qquad \ldots (8)$$

On comparing C.V.'s in (7) and (8), one can easily conclude that T' -scores, which assume variations in true mean μ, tend to have a larger error of measurement, i.e., tend to be less consistent as compared to T- scores which assumes μ to be constant.

Finally, for measuring the intensity or sensitivity of T- scores in statistical terms when μ varies randomly, we make use of the percentage increase in C.V.'s. Thus,

Sensitivity of T- scores = Percentage increase in C.V.

$$= \left[\frac{\text{C.V. for the updated distribution} - \text{C.V. for the basic distribution}}{\text{C.V. for the basic distribution}}\right] \times 100\,\%$$

$$= \left[\frac{11.11 - 10.00}{10.00}\right] \times 100\% = 11.1\%$$

TABLE – 1. T AND T' - SCORES FOR VARYING $\mu = E(\mu) = \theta$

FOR SOME FIXED VALUES OF $X = X'$

$\mu = E(\mu)$	$X = X'$ =35		$X = X'$ =40		$X = X'$ =45		$X = X'$ =55	
	T-Scores	T' - Scores	T-Scores	T' - Scores	T-Scores	T' - Scores	T-Scores	T' - Scores
35.0	50.00	50.00	62.50	60.00	75.00	70.00	100.00	90.00
37.5	43.75	45.00	56.25	55.00	68.75	65.00	93.75	85.00
40.0	37.50	40.00	50.00	50.00	62.50	60.00	87.50	80.00
42.5	31.25	35.00	43.75	45.00	56.25	55.00	81.25	75.00
45.0	25.00	30.00	37.50	40.00	50.00	50.00	75.00	70.00

47.5	18.75	25.00	31.25	35.00	43.75	45.00	68.75	65.00
50.0	12.50	20.00	25.00	30.00	37.50	40.00	62.50	60.00
52.5	6.25	15.00	18.75	25.00	31.25	35.00	56.25	55.00
55.0	0.00	10.00	12.50	20.00	25.00	30.00	50.00	50.00

TABLE – 2. T AND T' - SCORES FOR VARYING $X = X'$ FOR DATA SETS I & II

$X = X'$	DATA SET - I		DATA SET - II	
	T-Scores	T'-Scores	T-Scores	T'-Scores
35.0	37.5	30.0	37.5	50.0
37.5	43.75	35.0	43.75	55.0
40.0	50.0	40.0	50.0	60.0
42.5	56.25	45.0	56.25	65.0
45.0	62.5	50.0	62.5	70.0
47.5	68.75	55.0	68.75	75.0
50.0	75.0	60.0	75.0	80.0
52.2	81.25	65.0	81.25	85.0
55.0	87.5	70.0	87.5	90.0

Fig. - 1 : T - & T' - SCORES FOR DIFFERENT VALUES OF $\mu = E(\mu) = \theta$ FOR X = X' = 35

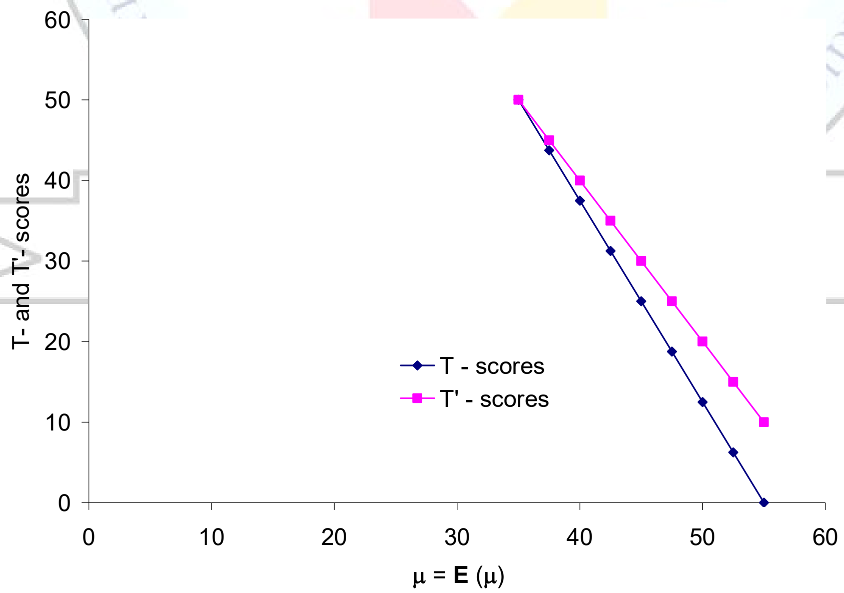


Fig. -2 : T - & T' - SCORES FOR DIFFERENT VALUES OF $\mu = E(\mu) = \theta$ FOR X = X' = 40

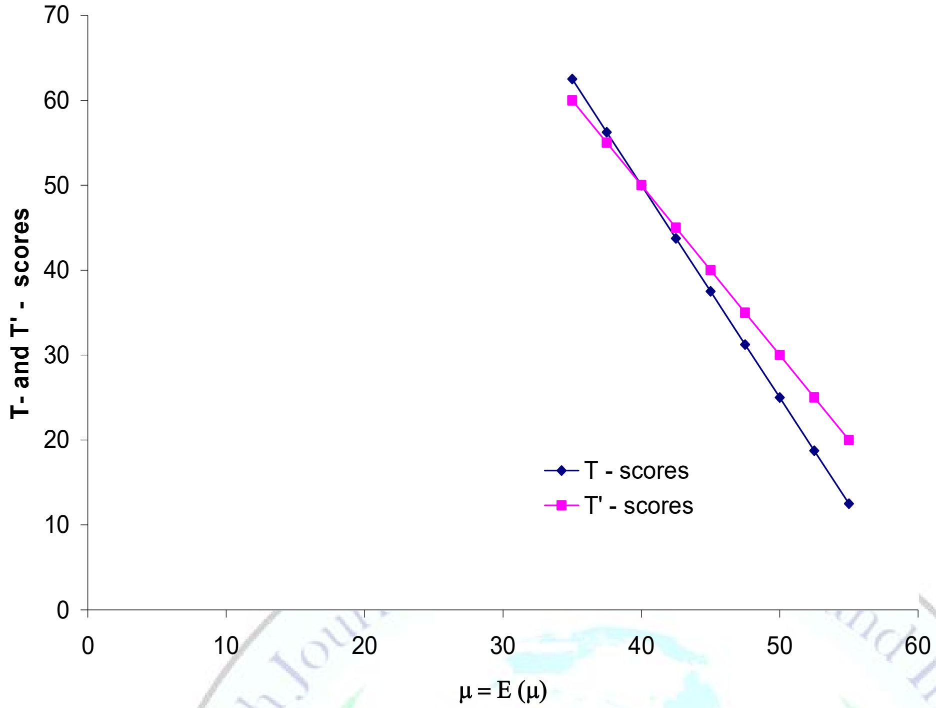


Fig. - 3 : T - & T' - SCORES FOR DIFFERENT VALUES OF $\mu = E(\mu) = \theta$ FOR X= X' = 45

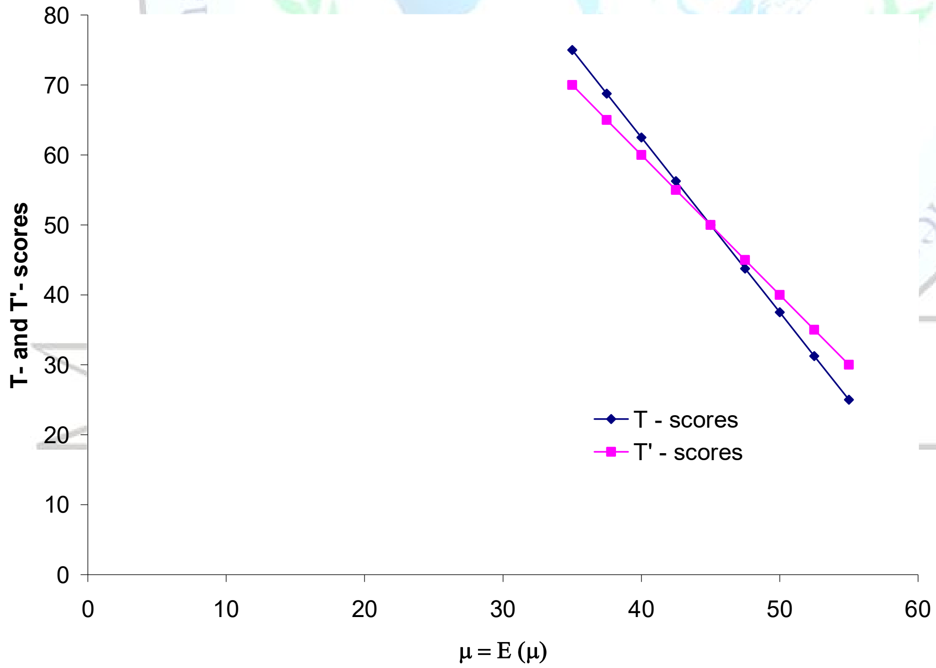


Fig. - 4 : T - & T' - SCORES FOR DIFFERENT VALUES OF μ = E(μ) = θ FOR X= X' = 55

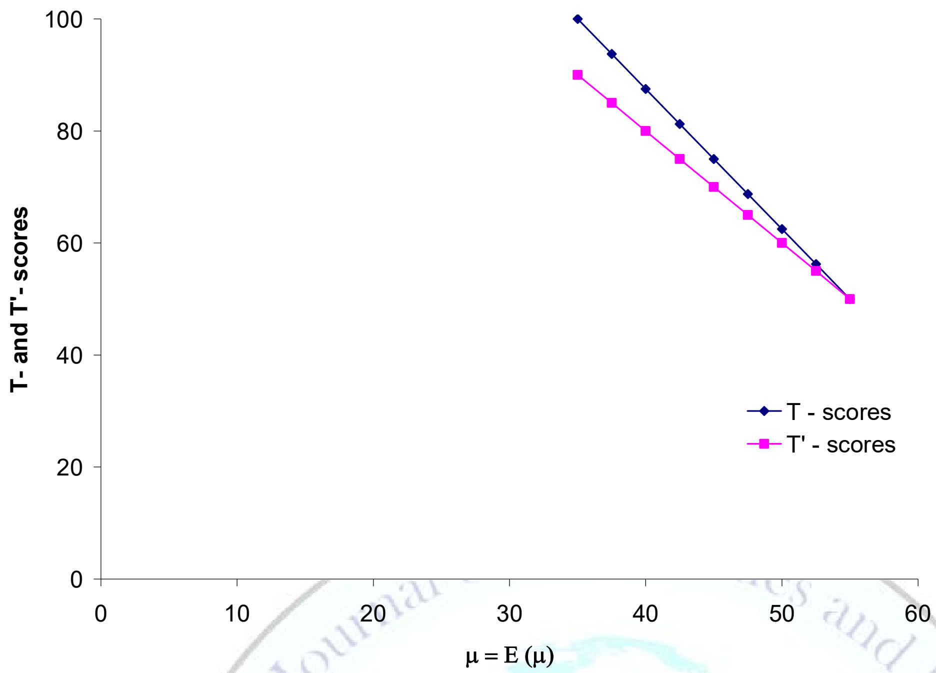


Fig. - 5 : T - & T' - SCORES FOR VARIOUS VALUES OF X = X' FOR DATA SET-I

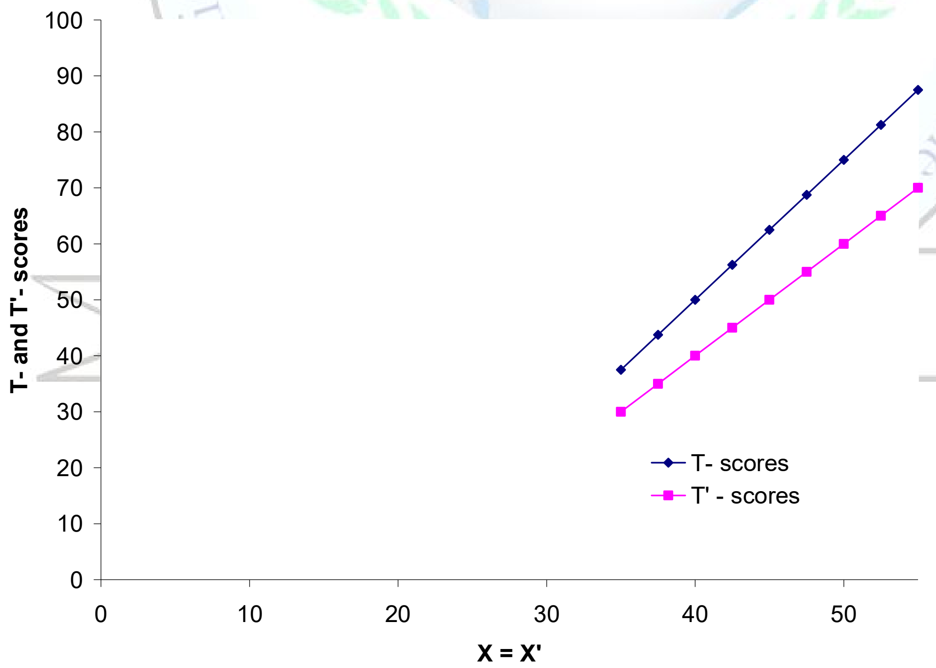
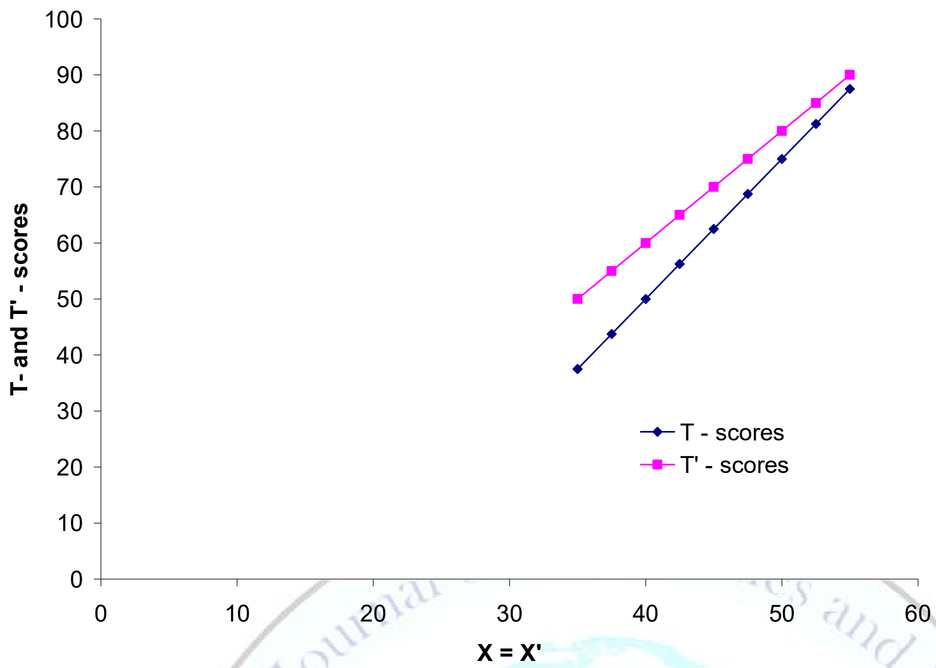


Fig. - 6 : T - & T' - SCORES FOR VARIOUS VALUES OF X = X' FOR DATA SET-II



8.0 Analysis:

The analysis of trends in T- and T' - in Table – 1 and also from Figs. 1 – 4, for I data set, we observe that –

(1) T- and T' - scores tend to differ as $\mu = \theta$ varies for some fixed values of $X = X'$. Obviously, T- scores are found to be sensitive or non-robust whenever μ varies randomly. Here,

(a) For $X = X' = \mu = \theta$, T- and T' - scores are equal to 50.

(b) For $X = X' < \mu = \theta$, T- and T' - scores are uniformly larger than T- scores.

(c) For $X = X' > \mu = \theta$, T- and T' - scores are uniformly smaller than T- scores.

(d) The measure of the sensitivity of T- scores, when μ varies randomly, amounts to 11.1% in the present case.

(2) The analysis of trends in T- and T' - in Table – 2 and also from Figs. 5 and 6, for both sets, we conclude that –

(a) For I data set in which $\mu < E(\mu) = \theta$, T'- scores are uniformly smaller than T- scores for varying values of $X = X'$.

(b) The trends for the II data set, i.e., for $\mu > E(\mu) = \theta$, get reversed and, in this case, T' - scores are uniformly greater than T- scores for varying values of $X = X'$.

Thus, the conclusion is that T- scores are found to be sensitive or non-robust in respect of variations in true scores and should be used cautiously whenever such variations in μ are suspected.

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