INTERNATIONAL RESEARCH JOURNAL OF HUMANITIES AND INTERDISCIPLINARY STUDIES
(Peer-reviewed, Refereed, Indexed & Open Access Journal)
DOI : 03.2021-11278686 ISSN : 2582-8568 IMPACT FACTOR : 6.865 (SJIF 2023)

FURTHER PROPERTIES OF K-FUNCTIONS IN CALCULUS

Yeong-Jeu Sun
Professor,
Department of Electrical Engineering,
I-Shou University, Kaohsiung (Taiwan)
E-mail: yjsun@isu.edu.tw

Sheng-Chieh Chen
Postgraduate Student,
Department of Electrical Engineering,
I-Shou University, Kaohsiung (Taiwan)

DOI No. 03.2021-11278686 **DOI Link ::** https://doi-ds.org/doilink/06.2023-73466626/IRJHIS2306019

Abstract:

Recently, the K-functions have been proposed by Sun et al; see [1] for details. This paper intends to do more in-depth research and derivation on the characteristics of the K-functions.

Keywords: *K-functions, definite integrals, differential equations*

Introduction:

As we know, there are many special functions in engineering and academia. With the research of more and more scholars on these special functions, various engineering applications are derived and created; see [1]-[8] and the references therein. In particular, the K-functions of the first type and the second type have been proposed and studied recently; see [1] for details. This paper will focus on such K-functions, exploring its various properties in differential and definite integral. Throughout this paper, some symbols are defined as follows: $N := \{1,2,3,\cdots\}$, $Z^+ := \{0,1,2,3,\cdots\}$, $P_m^n := \dfrac{n!}{(n-m)!}$, and $0! := 1$.

Problem formulation and main results

The definition of the K-functions is as follows

Definition 1[1]. The first type of K-functions is defined by

$$K_1(x,n) := x^n \cos x \text{, with } n \in Z^+. \tag{1}$$

Besides, the second type of K-functions is defined by

$$K_2(x,n) := x^n \sin x \text{, with } n \in Z^+ . \tag{2}$$

The related theorems about K-functions in indefinite integral are introduced as follows.

Lemma 1[1].

(i) $\int K_1(x,4k)dx = \sum_{i=0}^{k} P_{4i}^{4k} x^{4k-4i} \sin x + \sum_{i=0}^{k-1} \left(P_{4i+1}^{4k} x^{4k-4i-1} \cos x - P_{4i+2}^{4k} x^{4k-4i-2} \sin x \right.$

$\left. - P_{4i+3}^{4k} x^{4k-4i-3} \cos x \right) + C, \quad \forall\, k \in Z^+,$

(ii) $\int K_1(x,4k+1)dx = \sum_{i=0}^{k} \left(P_{4i}^{4k+1} x^{4k-4i+1} \sin x + P_{4i+1}^{4k+1} x^{4k-4i} \cos x \right)$

$- \sum_{i=0}^{k-1} \left(P_{4i+2}^{4k+1} x^{4k-4i-1} \sin x + P_{4i+3}^{4k+1} x^{4k-4i-2} \cos x \right) + C, \quad \forall\, k \in Z^+,$

(iii) $\int K_1(x,4k+2)dx = \sum_{i=0}^{k} \left(P_{4i}^{4k+2} x^{4k-4i+2} \sin x + P_{4i+1}^{4k+2} x^{4k-4i+1} \cos x - P_{4i+2}^{4k+2} x^{4k-4i} \sin x \right)$

$- \sum_{i=0}^{k-1} P_{4i+3}^{4k+2} x^{4k-4i-1} \cos x + C, \quad \forall\, k \in Z^+,$

(iv) $\int K_1(x,4k+3)dx = \sum_{i=0}^{k} \left(P_{4i}^{4k+3} x^{4k-4i+3} \sin x + P_{4i+1}^{4k+3} x^{4k-4i+2} \cos x \right.$

$\left. - P_{4i+2}^{4k+3} x^{4k-4i+1} \sin x - P_{4i+3}^{4k+3} x^{4k-4i} \cos x \right) + C, \quad \forall\, k \in Z^+,$

(v) $\int K_2(x,4k)dx = \sum_{i=0}^{k-1} \left(P_{4i+1}^{4k} x^{4k-4i-1} \sin x + P_{4i+2}^{4k} x^{4k-4i-2} \cos x - P_{4i+3}^{4k} x^{4k-4i-3} \sin x \right)$

$- \sum_{i=0}^{k} P_{4i}^{4k} x^{4k-4i} \cos x + C, \quad \forall\, k \in Z^+,$

(vi) $\int K_2(x,4k+1)dx = \sum_{i=0}^{k} \left(-P_{4i}^{4k+1} x^{4k-4i+1} \cos x + P_{4i+1}^{4k+1} x^{4k-4i} \sin x \right)$

$+ \sum_{i=0}^{k-1} \left(P_{4i+2}^{4k+1} x^{4k-4i-1} \cos x - P_{4i+3}^{4k+1} x^{4k-4i-2} \sin x \right) + C, \quad \forall\, k \in Z^+,$

(vii) $\int K_2(x,4k+2)dx = \sum_{i=0}^{k}\Big(-P_{4i}^{4k+2}x^{4k-4i+2}\cos x + P_{4i+1}^{4k+2}x^{4k-4i+1}\sin x$

$+P_{4i+2}^{4k+2}x^{4k-4i}\cos x\Big) - \sum_{i=0}^{k-1}P_{4i+3}^{4k+2}x^{4k-4i-1}\sin x + C, \quad \forall\, k \in Z^+,$

(viii) $\int K_2(x,4k+3)dx = \sum_{i=0}^{k}\Big(-P_{4i}^{4k+3}x^{4k-4i+3}\cos x + P_{4i+1}^{4k+3}x^{4k-4i+2}\sin x$

$+P_{4i+2}^{4k+3}x^{4k-4i+1}\cos x - P_{4i+3}^{4k+3}x^{4k-4i}\sin x\Big) + C, \quad \forall\, k \in Z^+.$

First, we propose novel properties of the K-functions.

Theorem 1.

(i) $\frac{d}{dx}\begin{bmatrix} K_1(x,n) \\ K_2(x,n) \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} K_1(x,n) \\ K_2(x,n) \end{bmatrix} + n\begin{bmatrix} K_1(x,n-1) \\ K_2(x,n-1) \end{bmatrix}, \quad \forall\, n \in N;$

(ii) $\left[\frac{dK_1(x,n)}{dx}\right]^2 + \left[\frac{dK_2(x,n)}{dx}\right]^2 = x^{2n-2}\left(n^2 + x^2\right), \quad \forall\, k \in Z^+;$

(iii) $\left[\frac{d^2K_1(x,n)}{dx^2}\right]^2 + \left[\frac{d^2K_2(x,n)}{dx^2}\right]^2 = x^{2n-4}\left[x^4 + \left(2n^2+2n\right)x^2 + n^2(n-1)^2\right], \quad \forall\, k \in Z^+.$

Proof. (i) and (ii)

$$\begin{cases} \dfrac{dK_1(x,n)}{dx} = nx^{n-1}\cos x - x^n \sin x, \\ \dfrac{dK_2(x,n)}{dx} = nx^{n-1}\sin x + x^n \cos x. \end{cases}$$

Thus, one has $\frac{d}{dx}\begin{bmatrix} K_1(x,n) \\ K_2(x,n) \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} K_1(x,n) \\ K_2(x,n) \end{bmatrix} + n\begin{bmatrix} K_1(x,n-1) \\ K_2(x,n-1) \end{bmatrix}$ and

$$\begin{aligned} &\left[\frac{dK_1(x,n)}{dx}\right]^2 + \left[\frac{dK_2(x,n)}{dx}\right]^2 \\ &= \left(nx^{n-1}\cos x - x^n\sin x\right)^2 + \left(nx^{n-1}\sin x + x^n\cos x\right)^2 \\ &= n^2x^{2n-2}\cos^2 x + x^{2n}\sin^2 x + n^2x^{2n-2}\sin^2 x + x^{2n}\cos^2 x \\ &= n^2x^{2n-2} + x^{2n} \\ &= x^{2n-2}\left(n^2 + x^2\right). \end{aligned}$$

(iii) $$\begin{cases} \dfrac{d^2K_1(x,n)}{dx^2} = n(n-1)x^{n-2}\cos x - 2nx^{n-1}\sin x - x^n\cos x; \\ \dfrac{d^2K_2(x,n)}{dx^2} = n(n-1)x^{n-2}\sin x + 2nx^{n-1}\cos x - x^n\sin x. \end{cases}$$

Obviously, we have

$$\left[\frac{d^2K_1(x,n)}{dx^2}\right]^2 + \left[\frac{d^2K_2(x,n)}{dx^2}\right]^2$$
$$= n^2(n-1)^2x^{2n-4} + 4n^2x^{2n-2} + x^{2n} - 2n(n-1)x^{2n-2}$$
$$= x^{2n-4}\left[x^4 + (2n^2+2n)x^2 + n^2(n-1)^2\right]$$

This completes the proof. □

So far, we propose related theorem about definite integrals of K-functions of the first type.

Theorem 2. For any $m \in N$ and $k \in Z^+$, one has

(i) $$\int_0^{m\pi} x^{2k}\cos x\,dx = \sum_{i=0}^{k-1}(-1)^{i+m}P_{2i+1}^{2k}(m\pi)^{2k-2i-1},$$

(ii) $$\int_0^{m\pi} x^{2k+1}\cos x\,dx = \left[(-1)^{k+m} - (-1)^k\right]\cdot P_{2k+1}^{2k+1} + \sum_{i=0}^{k-1}(-1)^{i+m}P_{2i+1}^{2k+1}(m\pi)^{2k-2i},$$

(iii) $$\int_0^{\frac{\pi(2m-1)}{2}} x^{2k}\cos x\,dx = \sum_{i=0}^{k}(-1)^{i+m+1}P_{2i}^{2k}\left[\frac{\pi(2m-1)}{2}\right]^{2k-2i},$$

(iv) $$\int_0^{\frac{\pi(2m-1)}{2}} x^{2k+1}\cos x\,dx = (-1)^{k+1}\cdot P_{2k+1}^{2k+1} + \sum_{i=0}^{k}(-1)^{i+m+1}P_{2i}^{2k+1}\left[\frac{\pi(2m-1)}{2}\right]^{2k-2i+1}.$$

Proof.

(i)

$$\int_0^{m\pi} x^{2k}\cos x\,dx = \left[\sum_{i=0}^{k-1}(-1)^i P_{2i+1}^{2k}\cdot x^{2k-2i-1}\right]\cdot\cos x\Big|_0^{m\pi}$$
$$= \sum_{i=0}^{k-1}(-1)^{i+m}P_{2i+1}^{2k}(m\pi)^{2k-2i-1},$$

in view of (i) and (iii) of Lemma 1.

(ii) By (ii) and (iv) of Lemma 1, it results

$$\int_0^{m\pi} x^{2k+1}\cos x\,dx = \left[(-1)^k \cdot P_{2k+1}^{2k+1}\cdot\cos x\right]\Big|_0^{m\pi} + \left[\sum_{i=0}^{k-1}(-1)^i\cdot P_{2i+1}^{2k+1}\cdot x^{2k-2i}\cdot\cos x\right]\Big|_0^{m\pi}$$

$$= \left[(-1)^{k+m}-(-1)^k\right]\cdot P_{2k+1}^{2k+1} + \sum_{i=0}^{k-1}(-1)^{i+m}\cdot P_{2i+1}^{2k+1}\cdot(m\pi)^{2k-2i}.$$

(iii) Owing (i) and (iii) of Lemma 1, we have

$$\int_0^{\frac{\pi(2m-1)}{2}} x^{2k}\cos x\,dx = \left[\sum_{i=0}^{k}(-1)^i P_{2i}^{2k}\cdot x^{2k-2i}\cdot\sin x\right]\Bigg|_{x=\frac{\pi(2m-1)}{2}}$$

$$= \sum_{i=0}^{k}(-1)^{i+m+1}\cdot P_{2i}^{2k}\cdot\left[\frac{\pi(2m-1)}{2}\right]^{2k-2i}.$$

(iv) It yields

$$\int_0^{\frac{\pi(2m-1)}{2}} x^{2k+1}\cos x\,dx = -(-1)^k\cdot P_{2k+1}^{2k+1} + \left[\sum_{i=0}^{k}(-1)^i P_{2i}^{2k+1}\cdot x^{2k-2i+1}\cdot\sin x.\right]\Bigg|_{x=\frac{\pi(2m-1)}{2}}$$

$$= (-1)^{k+1}\cdot P_{2k+1}^{2k+1} + \sum_{i=0}^{k}(-1)^{i+m+1}P_{2i}^{2k+1}\left[\frac{\pi(2m-1)}{2}\right]^{2k-2i+1},$$

in view of (ii) and (iv) of Lemma 1. This completes the proof.

Finally, we propose related theorem about definite integrals of K-functions of the second type.

Theorem 3. For any $m\in N$ and $k\in Z^+$, one has

(i) $$\int_0^{m\pi} x^{2k}\sin x\,dx = (-1)^k\cdot P_{2k}^{2k} + \sum_{i=0}^{k-1}(-1)^{i+m+1}\cdot P_{2i}^{2k}\cdot(m\pi)^{2k-2i},$$

(ii) $$\int_0^{m\pi} x^{2k+1}\sin x\,dx = \sum_{i=0}^{k}(-1)^{i+m+1}\cdot P_{2i}^{2k+1}\cdot(m\pi)^{2k-2i+1},$$

(iii) $$\int_0^{\frac{\pi(2m-1)}{2}} x^{2k}\sin x\,dx = (-1)^k\cdot P_{2k}^{2k} + \sum_{i=0}^{k-1}(-1)^{i+m+1}\cdot P_{2i+1}^{2k}\cdot\left[\frac{\pi(2m-1)}{2}\right]^{2k-2i-1},$$

(iv) $$\int_0^{\frac{\pi(2m-1)}{2}} x^{2k+1}\sin x\,dx=\sum_{i=0}^{k}(-1)^{i+m+1}\cdot P_{2i+1}^{2k+1}\cdot\left[\frac{\pi(2m-1)}{2}\right]^{2k-2i}.$$

Proof.

(i)

$$\int_0^{m\pi} x^{2k}\sin x\,dx=\sum_{i=0}^{k}(-1)^{i+1}\cdot P_{2i}^{2k}\cdot x^{2k-2i}\cdot\cos x\Big|_0^{m\pi}$$
$$=(-1)^k\cdot P_{2k}^{2k}+\sum_{i=0}^{k-1}(-1)^{i+m+1}\cdot P_{2i}^{2k}\cdot(m\pi)^{2k-2i},$$

in view of (v) and (vii) of Lemma 1.

(ii) From (vi) and (viii) of Lemma 1, it results

$$\int_0^{m\pi} x^{2k+1}\sin x\,dx=\sum_{i=0}^{k}(-1)^{i+1}\cdot P_{2i}^{2k+1}\cdot x^{2k-2i+1}\cdot\cos x\Big|_0^{m\pi}.$$
$$=\sum_{i=0}^{k}(-1)^{i+m+1}\cdot P_{2i}^{2k+1}\cdot(m\pi)^{2k-2i+1}.$$

(iii) Owing (v) and (vii) of Lemma 1, we have

$$\int_0^{\frac{\pi(2m-1)}{2}} x^{2k}\sin x\,dx=\left[\sum_{i=0}^{k}(-1)^{i+1}\cdot P_{2i}^{2k}\cdot x^{2k-2i}\cdot\cos x\right]_0^{\frac{\pi(2m-1)}{2}}$$
$$+\left[\sum_{i=0}^{k-1}(-1)^{i}\cdot P_{2i+1}^{2k}\cdot x^{2k-2i-1}\cdot\sin x\right]_0^{\frac{\pi(2m-1)}{2}}$$
$$=(-1)^k\cdot P_{2k}^{2k}+\sum_{i=0}^{k-1}(-1)^{i+m+1}\cdot P_{2i+1}^{2k}\cdot\left[\frac{\pi(2m-1)}{2}\right]^{2k-2i-1}.$$

(iv) It yields

$$\int_0^{\frac{\pi(2m-1)}{2}} x^{2k+1}\sin x\,dx=\sum_{i=0}^{k}(-1)^{i}\cdot P_{2i+1}^{2k+1}\cdot x^{2k-2i}\cdot\sin x\Big|_0^{\frac{\pi(2m-1)}{2}}$$
$$=\sum_{i=0}^{k}(-1)^{i+m+1}\cdot P_{2i+1}^{2k+1}\cdot\left[\frac{\pi(2m-1)}{2}\right]^{2k-2i},$$

in view of (vi) and (viii) of Lemma 1. This completes the proof. □

Examples: According to the results of Theorem 2 and Theorem 3, the following eight definite integrals can be easily obtained.

$$\int_0^{\pi} x^6 \cos x\, dx = -P_1^6\pi^5 + P_3^6\pi^3 - P_5^6\pi = -6\pi^5 + 120\pi^3 - 720\pi$$

$$\int_0^{\pi} x^7 \cos x\, dx = 2P_7^7 - P_1^7\pi^6 + P_3^7\pi^4 - P_5^7\pi^2 = 10080 - 7\pi^6 + 210\pi^4 - 2520\pi^2$$

$$\int_0^{\frac{\pi}{2}} x^6 \cos x\, dx = P_0^6\left(\frac{\pi}{2}\right)^6 - P_2^6\left(\frac{\pi}{2}\right)^4 + P_4^6\left(\frac{\pi}{2}\right)^2 - P_6^6 = \frac{\pi^6}{64} - \frac{15\pi^4}{8} + 90\pi^2 - 720$$

$$\int_0^{\frac{\pi}{2}} x^7 \cos x\, dx = P_7^7 + P_0^7\left(\frac{\pi}{2}\right)^7 - P_2^7\left(\frac{\pi}{2}\right)^5 + P_4^7\left(\frac{\pi}{2}\right)^3 - P_6^7\left(\frac{\pi}{2}\right)$$
$$= 5040 + \frac{\pi^7}{128} - \frac{21\pi^5}{16} + 105\pi^3 - 2520\pi$$

$$\int_0^{\pi} x^6 \sin x\, dx = -P_6^6 + P_0^6\pi^6 - P_2^6\pi^4 + P_4^6\pi^2 = -720 + \pi^6 - 30\pi^4 + 360\pi^2$$

$$\int_0^{\pi} x^7 \sin x\, dx = P_0^7\pi^7 - P_2^7\pi^5 + P_4^7\pi^3 - P_6^7\pi = \pi^7 - 42\pi^5 + 840\pi^3 - 5040\pi$$

$$\int_0^{\frac{\pi}{2}} x^6 \sin x\, dx = -P_6^6 + P_1^6\left(\frac{\pi}{2}\right)^5 - P_3^6\left(\frac{\pi}{2}\right)^3 + P_5^6\left(\frac{\pi}{2}\right) = -720 + \frac{3\pi^5}{16} - 15\pi^3 + 360\pi$$

$$\int_0^{\frac{\pi}{2}} x^7 \sin x\, dx = P_1^7\left(\frac{\pi}{2}\right)^6 - P_3^7\left(\frac{\pi}{2}\right)^4 + P_5^7\left(\frac{\pi}{2}\right)^2 - P_7^7 = \frac{7\pi^6}{64} - \frac{105\pi^4}{8} + 630\pi^2 - 5040$$

Conclusion:

In this paper, we have explored the relevant properties of the K-functions, and derive the differential equations and definite integral properties of the K-functions respectively. Finally, examples have also been given to verify the feasibility and effectiveness of the proposed main results.

References:

1. Y. J. Sun and S. C. Chen, "The Properties of Two Types of K-functions," Galaxy International

Interdisciplinary Research Journal, vol. 11, no. 2, pp. 17-22, 2023.

2. K. Ji, B. Zhang, T. Lu, and K. Ren, "Multi-Party Private Function Evaluation for RAM," IEEE Transactions on Information Forensics and Security, vol. 18, pp. 1252-1267, 2023.
3. Q. Zhang, J. Wu, H. Zhong, D. He, and J. Cui, "Efficient Anonymous Authentication Based on Physically Unclonable Function in Industrial Internet of Things," IEEE Transactions on Information Forensics and Security, vol. 18, pp. 233-247, 2023.
4. Z. Hajduk and G. R. Dec, "Very High Accuracy Hyperbolic Tangent Function Implementation in FPGAs," IEEE Access, vol. 11, pp. 23701-23713, 2023.
5. M. Ghita, C. Billiet, D. Copot, D. Verellen, and C. M. Ionescu, "Parameterisation of Respiratory Impedance in Lung Cancer Patients From Forced Oscillation Lung Function Test," IEEE Transactions on Biomedical Engineering, vol. 70, no. 5, pp. 1587-1598, 2023.
6. J. Liang and F. Tian, "An Online Algorithm for Virtualized Network Function Placement in Mobile Edge Industrial Internet of Things," IEEE Transactions on Industrial Informatics, vol. 19, no. 3, pp. 2496-2507, 2023.
7. Z. Y. Wang, X. P. Li, and H. C. So, "Robust Matrix Completion Based on Factorization and Truncated-Quadratic Loss Function," IEEE Transactions on Circuits and Systems for Video Technology, vol. 33, no. 4, pp. 1521-1534, 2023.
8. Z. Zhang, Y. Wang, J. Zhang, H. Zhang, Z. Ai, K. Liu, and F. Liu, "Asymptotic Stabilization Control of Fractional-Order Memristor-Based Neural Networks System via Combining Vector Lyapunov Function With M-Matrix," IEEE Transactions on Systems, Man, and Cybernetics: Systems, vol. 53, no. 3, pp. 1734-1747, 2023.